# Homework of Week 5

## Deadline: 9:00am, November 15(Thursday), 2018

1. (**Bonus score 5 points**) Prove the Poisson convergence theorem with weak dependence. Namely, for each $n$, suppose there are random variables $X_1^n, ..., X_n^n \in \{0,1\}$ such that

   - $\lim_{n\to\infty} \mathbb{E}[Y_n] = \lambda$ where $Y_n = \sum_{i=1}^n X_i^n$, and
   - For any $k$, $\lim_{n\to\infty} \sum_{1\le i_1<...<i_k\le n} \Pr\left(X_{i_1}^n = X_{i_2}^n = ... = X_{i_r}^n = 1\right) = \lambda^k/k!$

   Then $\lim_{n\to\infty} Y_n \sim Poi(\lambda)$, i.e. $\lim_{n\to\infty} \Pr(Y_n = k) = e^{-\lambda}\lambda^k/k!$ for any integer $k \ge 0$. (Hint: you may need Bonferroni inequalities)

2. The following problem models a simple distributed system wherein agents contend for resources but *back off* in the face of contention. Balls represent agents, and bins represent resources.

   The system evolves over rounds. Every round, balls are thrown independently and uniformly at random into $n$ bins. Any ball that lands in a bin by itself is served and removed from consideration. The remaining balls are thrown again in the next round. We begin with $n$ balls in the first round, and we will finish when every ball is served.

   - If there are $b$ balls at the start of a round, what is the expected number of balls at the start of the next round?
   - Suppose that every round the number of balls served was exactly the expected number of balls to be served. Show that all the balls would be served in $O(\ln\ln n)$ rounds. (Hint: If $x_j$ is the expected number of balls left after $j$ rounds, show and use that $x_{j+1} \le x_j^2/n$.)

3. Let $X_1, ..., X_n$ be independent and identically distributed Poisson random variables and $X = \sum_{i=1}^n X_i$. Let $\mathcal{E}$ be the event that all $X_i$'s are nonzero. Prove that $\Pr(\mathcal{E}|X = k)$ increases with $k$.

4. Let $X_1, ..., X_n$ be independent and identically distributed Poisson random variables and $X = \sum_{i=1}^n X_i$. Let $\mathcal{E}$ be the event that all $X_i$'s are nonzero. Prove that $\lim_{n\to\infty} \Pr(\mathcal{E}|X = m + \sqrt{2m\ln m}) - \Pr(\mathcal{E}|X = m - \sqrt{2m\ln m}) = 0$ where $m = n\ln n$.

5. Do Bernoulli experiment for 20 trials, using a new 1-Yuan coin. Record the result in a string $s_1 s_2 ... s_i ... s_{20}$, where $s_i$ is 1 if the $i^{th}$ trial gets Head, and otherwise is 0.