Homework of Week 12

This homework is optional. It will NOT affect your final score. If you choose to submit it , the deadline is 9:00am of January 10 (Thursday), 2019

1. Prove that the Markov property of a random process $\{X_n\}_{n=0}^{\infty}$ is equivalent to the following property: for any $n > k \geq 0$, any $I \subseteq \{0, ..., k-1\}$, and any $x_0, ..., x_k, x_n$, it holds that

$$\Pr(X_n = x_n | X_k = x_k, ...X_0 = x_0) = \Pr(X_n = x_n | X_k = x_k, X_i = x_i, i \in I).$$

2. Given a Markov chain $\{X_n\}_{n=0}^{\infty}$, prove that for any $n > 0$ and any states $x_0, ..., x_n$,

$$\Pr(X_n = x_n, X_{n-1} = x_{n-1}, ..., X_1 = x_1 | X_0 = x_0) = \prod_{i=1}^{n} \Pr(X_i = x_i | X_{i-1} = x_{i-1}).$$

3. We say a Markov chain is aperiodic if and only if all states in the chain are aperiodic. Given a finite-state aperiodic Markov chain, assume that each pair of states communicates. Then prove that if n is large enough, $p_{ij}^{(n)} > 0$ for all states i, j.

4. Given a Markov chain, let i and j be two states. Define $f_{ij}^{(k)}$ to be the probability that starting with state i at time 0, the chain first reaches state j at time k, and $p_{ij}^{(k)}$ be the k-step probability of reaching j from i. Prove that $p_{ij}^{(n)} = \sum_{i=1}^{n} f_{ij}^{(k)} p_{jj}^{(n-k)}$, where $p_{jj}^{(0)} = 1$.

5. Given a finite-state Markov chain, prove that
   (a) At least one state is recurrent.
   (b) (Not easy) All recurrent states are positive recurrent.