Homework of Week 11

Deadline: 9:00am, January 3 (Thursday), 2019

1. Use the Lovász Local Lemma to show that, if

$$4\binom{k}{2}\binom{n}{k-2}2^{1-\binom{k}{2}} \le 1,$$

   then it is possible to color the edges of K_n with two colors so that it has no monochromatic K_k subgraphs. Note that this is better than the result obtained by counting.

2. Let $G = (V, E)$ be an undirected graph and suppose each $v \in V$ is associated with a set $S(v)$ of $8r$ colors, where $r \geq 1$. Suppose, in addition, that for each $v \in V$ and $c \in S(v)$ there are at most r neighbors u of v such that c lies in $S(u)$. Prove that there is a coloring of G which assigns to each vertex v a color from $S(v)$ such that, for any edge $(u, v) \in E$, the colors assigned to u and v are different. [Let $A_{u,v,c}$ be the event that u and v are both colored with color c and apply the symmetric Lovász Local Lemma.]

3. Given $\beta > 0$, a vertex-coloring of a graph G is said to be β-frugal if (i) each pair of adjacent vertices has different colors, and (ii) no vertex has $\beta+1$ neighbors that have the same color. Prove that if G has maximum degree $\Delta \geq \beta^\beta$ with $\beta \geq 2$, then G has a β-frugal coloring with $16\Delta^{1+1/\beta}$ colors. [Hint: you may need to define two types of events corresponding to the two conditions of being β-frugal. Then an asymmetric Lovász Local Lemma can be used.]

4. Do Bernoulli experiment for 20 trials, using a new 1-Yuan coin. Record the result in a string $s_1 s_2 ... s_i ... s_{20}$, where s_i is 1 if the i^{th} trial gets Head, and otherwise is 0.