Probabilistic Method and Random Graphs

Lecture 1. Elementary probability theory with applications [1]

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[1] The slides are mainly based on Chapters 1 to 2 of *Probability and Computing*.

Important information

Course homepage

`http://z14120902.github.io/pm.html`

Teaching assistant

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Office hours: TBD

Homework

Submit in PDF to: prob_method@163.com
Deadline: 9:00am, Thursday

Grading policy

Homework+Participation: 50%
Final exam (Open book?): 50%

Warning: Enrolling in this course is at your own risk!

The brilliant history of probability theory

GAMBLERS	As long as human history?
Cardano	15xx, attempting to analyze games of chance Informal law of large numbers, sum of 3 dice
Fermat&Pascal	1654, fair division of the stake in an interrupted game of chance
Huygens	1657, comprehensive treatment, expectation values
Bernoulli	1713, *Ars Conjectandi*, a sound mathematical footing Law of large numbers
de Moivre	1718, *The Doctrine of Chances*, a sound mathematical footing
Gauss	18xx, application in astronomy, normal distribution
Laplace	1812, *Theorie analytique des probabilites*, fundamental results: MGF, MLS, hypothesis testing
...	
Kolmogorov	1933, *Foundations of the Theory of Probability*, modern axiomatic foundations
...	

Wisdom of probability theory

Laplace(1745-1827)
Probability theory is nothing but
a formulation of common sense

Advice from this book: Part of the research process in random processes is first to understand what is going on at a high level and then to use this understanding in order to develop formal mathematical proofs. ...To gain insight, you should perform experiments based on writing code to simulate the processes.

Why probability in CS: two fundamental ways

Algorithm design

- Randomized
- Probability-theory-based: statistical, derandomized $\cdots$
- Quantum computing

Algorithm analysis

- Average complexity
- Smoothed complexity: Spielman and Teng
- Learning theory

No probability, no viability!

Probability axioms and basic properties

A probability space (modeling a random process) has 3 elements

Sample space $\Omega \neq \emptyset$ The set of possible outcomes

Event family $\mathcal{F} \subseteq 2^{\Omega}$ The set of eligible events, a σ-algebra

Prob. function $\Pr : \mathcal{F} \to R$ The *likelihood* of the events

$\Pr$ satisfies 3 conditions:

- $Range(\Pr) \subseteq [0, 1]$
- $\Pr(\Omega) = 1$
- $\Pr(\bigcup_{i \geq 1} E_i) = \sum_{i \geq 1} \Pr(E_i)$ if the countably many events are mutually disjoint

Remarks

- We mainly consider the discrete case
- Events are sets, so Venn diagrams will be used for intuition

An example probability space

Coin flip

- $\Omega = \{H, T\}$
- $\mathcal{F} = 2^{\Omega}$
- $\Pr(H) = p, \Pr(T) = 1 - p$

$p = 1/2$ if the coin is unbiased.

Union bound

$$\Pr(E_1 \bigcup E_2) = \Pr(E_1) + \Pr(E_2) - \Pr(E_1 \bigcap E_2)$$

Inclusion-exclusion principle

$$\Pr(\bigcup_{i\geq 1}^{n} E_i) = \sum_{l=1}^{n} (-1)^{l-1} \sum_{i_1<i_2<\ldots<i_l} \Pr(\bigcap_{r=1}^{l} E_{i_r})$$

Union bound (Boole's Inequality)

$$\Pr(\bigcup_{i\geq 1} E_i) \leq \sum_{i\geq 1} \Pr(E_i)$$

Bonferroni Inequalities

- $\Pr(\bigcup_{i\geq 1}^{n} E_i) \leq \sum_{l=1}^{r} (-1)^{l-1} \sum_{i_1<i_2<\ldots<i_l} \Pr(\bigcap_{r=1}^{l} E_{i_r})$ for odd r
- $\Pr(\bigcup_{i\geq 1}^{n} E_i) \geq \sum_{l=1}^{r} (-1)^{l-1} \sum_{i_1<i_2<\ldots<i_l} \Pr(\bigcap_{r=1}^{l} E_{i_r})$ for even r

Independence and conditional probability

Definition: independent events

- $\Pr(E \bigcap F) = \Pr(E)\Pr(F)$
- Events $E_1, E_2, \ldots E_k$ are mutually independent if for any $I \subseteq [1,k]$, $\Pr(\bigcap_{i \in I} E_i) = \prod_{i \in I} \Pr(E_i)$

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Definition: conditional probability

- $\Pr(E|F) = \frac{\Pr(E \bigcap F)}{\Pr(F)}$, well-defined if $\Pr(F) \neq 0$
- Conditioning changes/restricts the sample space
- Probability changes when more information is available

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Corollary

- $\Pr(E|F) = \Pr(E)$ if E and F are independent
- Independence means that the probability of one event is not affected by the information on the other
- Chain rule: $\Pr(\bigcap_{i=1}^{n} A_i) = \prod_{i=1}^{n} \Pr(A_i | \bigcap_{j=1}^{i-1} A_j)$

Law of total probability

If $E_1, E_2, \ldots E_n$ are mutually disjoint and $\bigcup_{i=1}^{n} E_i = \Omega$, then $\Pr(B) = \sum_{i=1}^{n} \Pr(B \bigcap E_i) = \sum_{i=1}^{n} \Pr(B|E_i) \Pr(E_i)$.

Example

Find the probability that the sum of n dice is divisible by 6.

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Example

Find the probability that the sum of n dice is divisible by 6.

Solution:

- X_k: the result of the k-th roll for $1 \leq k \leq n$
- $Y_k = \sum_{i=1}^{k} X_i$ for $1 \leq k \leq n$
- $\Pr(Y_n \equiv 0 \mod 6) = \sum_{i=1}^{6} \Pr((Y_n \equiv 0 \mod 6) \cap (X_n = i))$
- Claim: $\Pr((Y_n \equiv 0 \mod 6) \cap (X_n = i))$
  $= \Pr(Y_{n-1} \equiv 6 - i \mod 6)\Pr(X_n = i)$

Basic laws

Law of total probability

If $E_1, E_2, ...E_n$ are mutually disjoint and $\bigcup_{i=1}^{n} E_i = \Omega$, then
$\Pr(B) = \sum_{i=1}^{n} \Pr(B \bigcap E_i) = \sum_{i=1}^{n} \Pr(B|E_i) \Pr(E_i)$.

Bayes'Law

If $E_1, E_2, ...E_n$ are mutually disjoint and $\bigcup_{i=1}^{n} E_i = \Omega$, then
$\Pr(E_j|B) = \frac{\Pr(B|E_j)\Pr(E_j)}{\Pr(B)} = \frac{\Pr(B|E_j)\Pr(E_j)}{\sum_{i=1}^{n}\Pr(B|E_i)\Pr(E_i)}$.

It is time to solve a BIG Problem!

- Monty Hall problem
- First appeared at *Ask Marilyn* column of Parade, 9.9.1990
- See the demo
- Named after the celebrated TV host Monty Hall
- Confusing, so that formal proofs are not convincing (Paul Erdos & Andrew Vazsonyi)
- What's your answer?

Marilyn in 2017

Monty in 1970'

Solution to Monty Hall problem

Proof

- Reference for a formal proof: The Monty Hall Problem, by Afra Zomorodian, 1998
- An intuitive proof: keeping for one door but switching for two

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God is fair: smart Miss Marilyn made silly mistakes

- January 22, 2012: How likely are you chosen over one year?
- May 5, 2013: How many 4-digit briefcase combinations contain a particular digit?
- June 22, 2014: How many work hours is necessary?
  6 together, but a 4-hour gap for each
- January 25, 2015: Which salary options do you prefer?
  Annual $1000 or semi-annual $300 raises

Random variables and expectation

Random variable

- A real-valued function on the sample space of a probability space, $X : \Omega \rightarrow R$
- Random variables on this same probability space have both functional operations and probability operations

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Probability of a random variable

- $X = a$ stands for the event $\{s \in \Omega | X(s) = a\}$
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Independent random variables

- $\Pr((X = x) \bigcap (Y = y)) = \Pr(X = x) \Pr(Y = y)$
- Gengerally, $\Pr(\bigcap_{i \in I} (X_i = x_i)) = \prod_{i \in I} \Pr(X_i = x_i)$ for any I

Expectation: a basic characteristic

Definition

- $\mathbb{E}[X] = \sum_{i \in Range(X)} i * \Pr(X = i)$
- It's finite if $\sum_{i \in Range(X)} |i| * \Pr(X = i)$ converges

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Linearity of expectation

- $\mathbb{E}[\sum_{i=1}^{n} a_i X_i] = \sum_{i=1}^{n} a_i \mathbb{E}[X_i]$
- No independence is required
- The only condition is that each $\mathbb{E}[X_i]$ is bounded
- The most important property of expectation!

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Product Counterpart

$\mathbb{E}[X * Y] = \mathbb{E}[X]\mathbb{E}[Y]$ if they are **independent**.

Bernoulli distribution

Bernoulli random variable

- $\Pr(X = 1) = p, \Pr(X = 0) = 1 - p$
- Modeling coin flip
- $\mathbb{E}[X] = p * 1 + (1 - p) * 0 = p$
- $X^k = X$

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An example

How many triangles among 4 nodes when the links appear independently randomly?

Binomial distribution

Binomial random variable

- The number of successes in n independent trials of the Bernoulli experiment with success probability p
- For any $0 \le i \le n$, $\Pr(X = i) = C_n^i p^i (1-p)^{n-i}$
- $X = \sum_{i=1}^{n} X_i$
- $\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X_i] = np$

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Application

Router packets sampling

Geometric distribution

The story of Farmer&Rabbit

Geometric distribution

The story of Farmer&Rabbit

Geometric random variable

- The number of independent trials until success, where each trial has success probability p
- $\Pr(X = i) = (1-p)^{i-1}p$ for $i \geq 1$
- $\mathbb{E}[X] = \sum_{i \geq 1} i(1-p)^{i-1}p = 1/p$

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The story of Farmer&Rabbit

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Memoryless: particular to geometric distribution

For geometric random variable X, if $n > 0$,
$Pr(X = n + k | X > k) = Pr(X = n)$

Application: coupon collector's problem

Problem statement

The # of boxes of milk to buy before you have all coupon types?

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- X_i: the number of boxes bought while you have $i-1$ types of coupons until you get the ith type
- $X = \sum_{i=1}^{n} X_i$
- X_i: geometric random variable with parameter $p_i = 1 - \frac{i-1}{n}$
- $\mathbb{E}[X_i] = \frac{1}{p_i} = \frac{n}{n-i+1}$

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$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X_i] = n \sum_{i=1}^{n} \frac{1}{i} = n \ln n + \Theta(n).$$

Thanks!