SOLUTION MANUAL FOR *PROBABILISTIC METHOD AND RANDOM GRAPHS*

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ABSTRACT. In this note, we present solutions for selected homework problems for course *Probabilistic method and random graphs*: http://z14120902.github.io/pm.html.

CONTENTS

1. HW1

1. Memoryless implies

$$\forall m,n \in \mathbb{Z}^+ : \mathbf{P}(X > m+n \mid X > m) = \mathbf{P}(X > n)$$

or

$$\forall m,n \in \mathbb{Z}^+ : \frac{\mathbf{P}(X > m+n \cap X > m)}{\mathbf{P}(X > m)} = \frac{\mathbf{P}(X > m+n)}{\mathbf{P}(X > m)} = \mathbf{P}(X > n)$$

or

$$\forall m,n \in \mathbb{Z}^+ : \mathbf{P}(X > m+n) = \mathbf{P}(X > m)\mathbf{P}(X > n)$$

the rest will be easy.

2.

1. Recall *Coupon collector's problem*, there are 2 coupons in this case

$$\mathbf{E}X = 1 + 2 = 3$$

2.

- after first kid, success within 5 kids: $\sum_{i=1}^{4} \frac{i}{2^i} = \frac{13}{8}$
- after first kid, fail within 5 kids: $4\frac{1}{2^4} = \frac{1}{4}$

$\mathbf{E}X = 1 + \frac{13}{8} + \frac{1}{4} = \frac{23}{8}$

2. HW2

1.

$$\mathbf{P}\left(X \geq (1+\delta)\mu_H\right) = \mathbf{P}\left(e^{\lambda X} \geq e^{\lambda(1+\delta)\mu_H}\right) \leq \frac{\mathbf{E}[e^{\lambda X}]}{e^{\lambda(1+\delta)\mu_H}} = \frac{\mathbf{E}[e^{\lambda \sum_{i=1}^n X_i}]}{e^{\lambda(1+\delta)\mu_H}} = \frac{e^{\sum_{i=1}^n \left(p_i e^{\lambda} + (1-p_i)\right)}}{e^{\lambda(1+\delta)\mu_H}} \leq \frac{e^{\sum_{i=1}^n p_i\left(e^{\lambda}-1\right)}}{e^{\lambda(1+\delta)\mu_H}} = \frac{e^{\mu\left(e^{\lambda}-1\right)}}{e^{\lambda(1+\delta)\mu_H}} \leq \frac{e^{\mu_H\left(e^{\lambda}-1\right)}}{e^{\lambda(1+\delta)\mu_H}} = \left(\frac{e^{\left(e^{\lambda}-1\right)}}{e^{\lambda(1+\delta)}}\right)^{\mu_H}$$

the rest will be easy.

2. we need the following

$$\mathbf{E}[e^{\lambda a_i X_i}] = p_i e^{\lambda a_i} + 1 - p_i = 1 + p_i\left(e^{\lambda a_i} - 1\right) \leq \underline{e^{p_i\left(e^{\lambda a_i}-1\right)} \leq e^{p_i a_i\left(e^{\lambda}-1\right)}}$$

or

$$e^{\lambda a_i} - 1 \leq a_i\left(e^{\lambda} - 1\right)$$

or

$$\frac{e^{\lambda a_i} - 1}{a_i} \leq \frac{e^{\lambda * 1} - 1}{1}$$

this is slope of line through $\left(x, e^{\lambda x}\right)$ and $(0,1)$, which is obvious via plot of function $e^{\lambda x}$, the rest will be easy.

3.

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1.

$$\mathbf{E}f(Z) = \sum_i p_i f(z_i) = \sum p_i f\left(z_i * 1 + (1-z_i) * 0\right) \le \sum_i p_i z_i f(1) + p_i (1-z_i) f(0) = pf(1) + (1-p)f(0) = \mathbf{E}f(X)$$

2. Omit

3. HW4

1. the exact probability is

$$\mathbf{P}(n\ bins\ m\ balls\ max\ load = 1) = \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{m-1}{n}\right)$$

1. we need $\mathbf{P}(n\ bins\ m\ balls\ max\ load = 1) \le \frac{1}{e}$ or

$$\mathbf{P}(n\ bins\ m\ balls\ max\ load = 1) \le e^{-\frac{1}{n}} e^{-\frac{2}{n}} \dots e^{-\frac{m-1}{n}} = e^{-\frac{m(m-1)}{2n}} \le \frac{1}{e}$$

we can calculate m

2. we need $\mathbf{P}(n\ bins\ m\ balls\ max\ load = 1) \ge \frac{1}{2}$ or

$$\mathbf{P}(n\ bins\ m\ balls\ max\ load = 1) \ge e^{-\frac{1}{n}-\frac{1}{n^2}} e^{-\frac{2}{n}-\frac{2^2}{n^2}} \dots e^{-\frac{m-1}{n}-\frac{(m-1)^2}{n^2}} = e^{-\frac{m(m-1)}{2n}-\frac{(m-1)m(2m-1)}{6n^2}} \ge \frac{1}{2}$$

we can calculate m

2.

1. $\mathbf{P}(X=n) = e^{-\mu}\frac{\mu^n}{n!}$

$$\mathbf{P}(Y=k) = \sum_{n=k}^{\infty} \mathbf{P}(X=n)\binom{n}{k} p^k (1-p)^{n-k} = \sum_{n=k}^{\infty} e^{-\mu}\frac{\mu^n}{n!}\binom{n}{k} p^k (1-p)^{n-k}$$

$$= \frac{e^{-\mu}p^k}{k!}\sum_{n=k}^{\infty}\frac{\mu^n (1-p)^{n-k}}{(n-k)!} = \frac{e^{-\mu}(\mu p)^k}{k!}\sum_{n=k}^{\infty}\frac{\mu^{n-k}(1-p)^{n-k}}{(n-k)!} = \frac{e^{-\mu}(\mu p)^k}{k!} e^{\mu(1-p)} = e^{-\mu p}\frac{(\mu p)^k}{k!}$$

Z can be proved likewise

2.

$$\mathbf{P}(Y=k_1, Z=k_2) = \mathbf{P}(X=k_1+k_2)\binom{k_1+k_2}{k_1} p^{k_1}(1-p)^{k_2} = e^{-\mu}\frac{\mu^{k_1+k_2}}{(k_1+k_2)!}\frac{(k_1+k_2)!}{k_1!k_2!} p^{k_1}(1-p)^{k_2} = e^{-\mu p}\frac{(\mu p)^{k_1}}{k_1!} e^{-\mu(1-p)}\frac{(\mu(1-p))^{k_2}}{k_2!} = \mathbf{P}(Y=k_1)\mathbf{P}(Z=k_2)$$

3.

1. **for** $n=1$, there are no other students

$$\mathbf{P}(2\ students\ same\ birthday) = 0$$

for $n \in \{2, \dots, 365\}$
the probability of max load is 1 is

$$\mathbf{P}(max\ load = 1) = \left(1-\frac{1}{365}\right)\left(1-\frac{2}{365}\right)\dots\left(1-\frac{n-1}{365}\right)$$

thus

$$\mathbf{P}(2\ students\ same\ birthday) = 1-\left(1-\frac{1}{365}\right)\left(1-\frac{2}{365}\right)\dots\left(1-\frac{n-1}{365}\right)$$

for $n \in \{365, 366, \dots\}$

$$\mathbf{P}(2\ students\ same\ birthday) = 1$$

2. **for** $n=1$, there are no other students

$$\mathbf{P}(existing\ another\ students\ same\ birthday) = 0$$

for $n \in \{2, \dots, 365\}$

$$\mathbf{P}(no\ other\ students\ same\ birthday) = \left(\frac{364}{365}\right)^{n-1}$$

$$\mathbf{P}(existing\ another\ students\ same\ birthday) = 1-\left(\frac{364}{365}\right)^{n-1}$$

4. $\mathbf{P}(X \ge x) \le \frac{\mathbf{E}e^{\lambda X}}{e^{\lambda x}} = \frac{e^{\mu(e^x-1)}}{e^{\lambda x}}$. let $\lambda = \ln\left(\frac{x}{\mu}\right)$

4. HW5

1. This is called Brun's sieve, Alon's book chapter 8.3.

2.

1. probability of a bin with load 1

$$\mathbf{P}(X_i = 1) = \binom{b}{1}\frac{1}{n}\left(1-\frac{1}{n}\right)^{b-1}$$

the expected balls will be served

$$\mathbf{E}X = n\mathbf{P}(X_i = 1) = b\left(1-\frac{1}{n}\right)^{b-1}$$

thus, expected number of balls at the start of the next round $b - b\left(1-\frac{1}{n}\right)^{b-1}$

2.

$$x_{j+1} = x_j - x_j\left(1-\frac{1}{n}\right)^{x_j-1} = x_j\left[1-\left(1-\frac{1}{n}\right)^{x_j-1}\right]$$

consider $f(x) = \left(1-\frac{1}{n}\right)^{x+1} - \left(1-\frac{x}{n}\right)$, we can get $x_{j+1} \le \frac{x_j^2}{n}$ or $\ln x_{j+1} \le 2\ln x_j - \ln n$ the rest will be easy.

3. Recall *Poisson Approximation Theorem*. $\mathbf{P}(X_1 \ne 0 \cap \cdots \cap X_n \ne 0 \mid \sum X_i = k)$ is the same probability of all bins are not empty in k balls into n bins model, the probability increases with k for the obvious reason.

4. Recall *Poisson Approximation Theorem*. $\lim_{n\to\infty}\mathbf{P}\left(\mathscr{E}|X = m+\sqrt{2m\ln m}\right) - \mathbf{P}\left(\mathscr{E}|X = m-\sqrt{2m\ln m}\right) \le$ the probability of at least one empty bin after throwing $m-\sqrt{2m\ln m}$ balls but at least one among the next $2\sqrt{2m\ln m}$ balls goes into that bin

$$\le \lim_{n\to\infty}\frac{2\sqrt{2m\ln m}}{n} = \lim_{n\to\infty}\frac{2\sqrt{2n\ln n\ln(n\ln n)}}{n} = \lim_{n\to\infty}\frac{2\sqrt{2n(\ln n)^2 + 2n\ln n\ln\ln n}}{n} \sim \lim_{n\to\infty}\frac{\ln n}{\sqrt{n}} \to 0$$

5. HW6

1.

1. $m = n, \lambda = 1$: $e\sqrt{n}\left(e^{-1}\frac{1^1}{1!}\right)^n \le \sqrt{n}e^{1-n}$

2. $\frac{n!}{n^n}$

2. [Mitzenmacher and Upfal, 2005] Theorem 5.10

6. HW7

1. for any graph G over n vertices, we need to prove for model $\mathscr{G}_n$ and $\mathscr{G}_{n,\frac{1}{2}}$

$$\mathbf{P}_{\mathscr{G}_n}(G) = \mathbf{P}_{\mathscr{G}_{n,\frac{1}{2}}}(G) = \frac{1}{2^{\binom{n}{2}}}$$

2. [Erdos and R&WI, 1959] Theorem 1

7. HW8

1. we need

$$p = \mathbf{P}\left(|S| \ge \frac{|V|}{D+1}\right) \ge \frac{1}{2D|V|^2}$$

since

$$\mathbf{E}|S| \ge \sum_{i=1}^{n}\frac{1}{d_i+1} \ge \frac{|V|}{D+1}$$

$$\mathbf{E}|S| = \sum_{|S|\ge\frac{|V|}{D+1}}|S|\mathbf{P}(|S|) + \sum_{|S|<\frac{|V|}{D+1}}|S|\mathbf{P}(|S|) \le |V|p + \left(\frac{|V|}{D+1}-1\right)(1-p)$$

thus

$$|V|p + \left(\frac{|V|}{D+1}-1\right)(1-p) \ge \frac{|V|}{D+1}$$

thus

$$p\left(\frac{D|V|}{D+1}+1\right) \ge 1$$

thus

$$p \ge \frac{D+1}{D+1+D|V|} > \frac{1}{2D|V|^2}$$

2.

1. It is an independent set because of the construction process: for each vertex i, $i \in S(\sigma)$ if and only if no neighbor j of i precedes i in the permutation σ. For vertex i and its neighbors $\Gamma(i)$, only one of them can be chosen.

2. random permutation

3.

1. consider 2-color edges of K_n

$$\mathbf{P}(\textit{a 4-clique is monochromatic}) = 2 * \frac{1}{2^{\binom{4}{2}}} = 2^{-5}$$

thus

$$\textit{total number of monochromatic copies of } K_4 \leq \binom{n}{4} 2^{-5}$$

2. randomly assign color to edges

8. HW9

1.

$$\mathbf{E}[\textit{total number of monochromatic copies of } K_k \textit{ for 2-coloring the edges of } K_n] = 2 * \binom{n}{k} \frac{1}{2^{\binom{k}{2}}} = \binom{n}{k} 2^{1-\binom{k}{2}}$$

we remove 1 vertex from each above monochromatic copies of K_k, we will have a $K_x : x = n - \binom{n}{k} 2^{1-\binom{k}{2}}$. And there are no monochromatic copies of K_k.

2. Use the derandomization techinique mentioned in the class. Choose each assignment that makes total number of monochromatic copies below its expectation.

Algorithm 1 find monochromatic K_4

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edges $:= x_1, x_2, \ldots, x_m$
color choices $:= v_k \in \{0,1\}$
for $k = 1$ **to** m **do**

$$x_k = \operatorname*{arg\,min}_{v_k \in \{0,1\}} \mathbf{E}\left[\textit{total number of monochromatic } K_4 \mid x_1 = v_1, \ldots, x_{k-1} = v_{k-1}, x_k = v_k\right]$$

end for

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3. similar idea

Algorithm 2 permutation σ

`vertices` $V := \{v_1, v_2, \ldots, v_n\}$
`permutated vertices` $X := \{x_1, x_2, \ldots, x_n\}$
for $k = 1$ **to** n **do**

$$x_k = \operatorname*{arg\,max}_{v_k \in V \setminus \{v_1, v_2, \ldots, v_{k-1}, N(v_1), N(v_2), \ldots, N(v_{k-1})\}} \mathbb{E}\left[S(\sigma) | x_1 = v_1, \ldots, x_{k-1} = v_{k-1}, x_k = v_k\right]$$

end for

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9. HW10

1. https://people.math.osu.edu/nguyen.1261/6501/Note-random1.pdf

2. Discussed in class. One of the difficulties might be random variables are not independent.

3. A direct applicaiton of LLL. $p = \frac{2}{2^r}$, $d \leq 2^{r-3}$. $4pd = 4\frac{2}{2^r}d \leq 4\frac{2}{2^r}2^{r-3} = 1$

10. HW11

1. A direct applicaiton of LLL. $p = 2^{1-\binom{k}{2}}$, $d \leq \binom{k}{2}\binom{n}{k-2}$. $4pd = 4\binom{k}{2}\binom{n}{k-2}2^{1-\binom{k}{2}}$

2. A direct applicaiton of LLL. $p = \frac{1}{8r}$, $d \leq 2(r-1)$. $4pd = 4\frac{1}{8r}2(r-1) < 1$

3. https://people.eecs.berkeley.edu/~sinclair/cs271/n24.pdf Theorem 24.11

References

[Erdos and R&WI, 1959] Erdos, P. and R&WI, A. (1959). On random graphs i. *Publ. Math. Debrecen*, 6:290–297.
[Mitzenmacher and Upfal, 2005] Mitzenmacher, M. and Upfal, E. (2005). *Probability and computing: Randomized algorithms and probabilistic analysis*. Cambridge university press.

11. KEY POINTS FOR FINAL

- Chernoff bound technique: Moment-generating function + Markov inequlity
- bin and ball model
- Union bound
- 2 forms of local lemma: $4pd$, $x_i \prod_{j \in \Gamma(i)} (1 - x_j)$