EQUIVALENCE OF PROBABILITY TAIL BOUND AND LAPLACE TRANSFORM DECAY FOR SUBGAUSSIAN VARIABLES

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Definition 1 (Subgaussian random variable)**.** *A real-valued random variable X is σ-subgaussian if*

$$\exists\sigma > 0, \forall s \in \mathbb{R} : \log \mathbf{E} e^{sX} \leq \frac{s^2\sigma^2}{2}$$

Claim 2. *subgaussian tail bound implies Laplace transform decay.*

Proof. Given subgaussian random variable X

$$\log \mathbf{E} e^{sX} \leq \frac{s^2\sigma^2}{2}$$

By Markov inequity

$$\Pr\left(f - \mathbf{E}f \geq t\right) \leq \inf_{s>0} \frac{\mathbf{E}e^{s(f-\mathbf{E}f)}}{e^{st}} = \inf_{s>0} e^{\log \mathbf{E}e^{s(f-\mathbf{E}f)} - st} \leq \inf_{s>0} e^{\frac{s^2\sigma^2}{2} - st} = e^{-\frac{t^2}{2\sigma^2}}$$

□

Claim 3. *Laplace transform decay implies subgaussian tail bound.*

Proof. Suppose we have a random variable with Laplace transform decay

$$\exists c > 0 : \Pr\left(|X - \mathbf{E}X| \geq t\right) \leq 2e^{-ct^2}$$

$\forall a \in (0, c)$

$$\begin{aligned}
& \mathbf{E}e^{a(X-\mathbf{E}X)^2} \\
= \; & 1 + \mathbf{E}\textstyle\int_0^{|X-\mathbf{E}X|} d\left(e^{at^2}\right) \\
= \; & 1 + \mathbf{E}\textstyle\int_0^{|X-\mathbf{E}X|} 2ate^{at^2} dt \\
= \; & 1 + \mathbf{E}\textstyle\int_0^{\infty} 2ate^{at^2} \mathbf{1}\left(|X - \mathbf{E}X| > t\right) dt \\
= \; & 1 + \textstyle\int_0^{\infty} 2ate^{at^2} \Pr\left(|X - \mathbf{E}X| > t\right) dt \\
\leq \; & 1 + \textstyle\int_0^{\infty} 2ate^{at^2} 2e^{-ct^2} dt \\
= \; & 1 + 4a\textstyle\int_0^{\infty} te^{-(c-a)t^2} dt \\
= \; & 1 + \tfrac{2a}{-(c-a)} \textstyle\int_0^{\infty} e^{-(c-a)t^2} d\left(-\left(c-a\right)t^2\right) \\
= \; & 1 + \tfrac{2a}{-(c-a)} [e^{-(c-a)t^2} \;|_0^{\infty}] \\
= \; & 1 + \tfrac{2a}{c-a} \\
= \; & \tfrac{c+a}{c-a}
\end{aligned}$$

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$$\begin{aligned}
&\mathbf{E}e^{s(X-\mathbf{E}X)}\\
=\;& 1+\mathbf{E}\textstyle\sum_{i\geq 0}\frac{(s(X-\mathbf{E}X))^{i+2}}{(i+2)!}\\
=\;& 1+\mathbf{E}\textstyle\sum_{i\geq 0}s^2\left(X-\mathbf{E}X\right)^2\frac{(s(X-\mathbf{E}X))^{i}}{(i+2)!}\\
=\;& 1+\mathbf{E}\textstyle\sum_{i\geq 0}s^2\left(X-\mathbf{E}X\right)^2\frac{(s(X-\mathbf{E}X))^{i}}{i!}\left(\frac{1}{i+1}-\frac{1}{i+2}\right)\\
=\;& 1+\mathbf{E}\textstyle\sum_{i\geq 0}s^2\left(X-\mathbf{E}X\right)^2\frac{(s(X-\mathbf{E}X))^{i}}{i!}\int_0^1\left(y^i-y^{i+1}\right)dy\\
=\;& 1+\textstyle\int_0^1\left(1-y\right)\mathbf{E}[s^2\left(X-\mathbf{E}X\right)^2\sum_{i=0}\frac{(s(X-\mathbf{E}X))^{i}}{i!}y^i]dy\\
=\;& 1+\textstyle\int_0^1\left(1-y\right)\mathbf{E}[s^2\left(X-\mathbf{E}X\right)^2e^{ys(X-\mathbf{E}X)}]dy\\
\leq\;& 1+\textstyle\int_0^1\left(1-y\right)\mathbf{E}[s^2\left(X-\mathbf{E}X\right)^2e^{|s(X-\mathbf{E}X)|}]dy\\
\leq\;& 1+\tfrac{s^2}{2}\mathbf{E}[(X-\mathbf{E}X)^2e^{\frac{s^2}{2a}+\frac{a(X-\mathbf{E}X)^2}{2}}]\\
=\;& 1+\tfrac{s^2}{2}\mathbf{E}[(X-\mathbf{E}X)^2e^{\frac{s^2}{2a}}e^{\frac{a(X-\mathbf{E}X)^2}{2}}]\\
=\;& 1+\tfrac{s^2}{2}e^{\frac{s^2}{2a}}\mathbf{E}[(X-\mathbf{E}X)^2e^{\frac{a(X-\mathbf{E}X)^2}{2}}]\\
=\;& 1+\tfrac{s^2}{a}e^{\frac{s^2}{2a}}\mathbf{E}[\tfrac{a(X-\mathbf{E}X)^2}{2}e^{\frac{a(X-\mathbf{E}X)^2}{2}}]\\
\leq\;& 1+\tfrac{s^2}{a}e^{\frac{s^2}{2a}}\mathbf{E}[e^{a(X-\mathbf{E}X)^2}]\\
\leq\;& 1+\tfrac{(c+a)s^2}{(c-a)a}e^{\frac{s^2}{2a}}\\
\leq\;& \left(1+\tfrac{(c+a)s^2}{(c-a)a}\right)e^{\frac{s^2}{2a}}\\
\leq\;& e^{\frac{(c+a)s^2}{(c-a)a}}e^{\frac{s^2}{2a}}\\
\leq\;& e^{\frac{a+3c}{2a(c-a)}s^2}
\end{aligned}$$

let $t=\frac{a}{c}\in(0,1)$

$$\mathbf{E}e^{s(X-\mathbf{E}X)}\leq\inf_t e^{\frac{t+3}{t(1-t)}\frac{s^2}{2c}}$$

$\min f\left(t\right)=\frac{t+3}{t(1-t)}$ s.t. $t\in(0,1)$

$$f'=\frac{t\left(1-t\right)-\left(1-2t\right)\left(t+3\right)}{t^2\left(1-t\right)^2}=\frac{t^2+6t-3}{t^2\left(1-t\right)^2}=\frac{\left(t-\left(-3+2\sqrt{3}\right)\right)\left(t-\left(-3-2\sqrt{3}\right)\right)}{t^2\left(1-t\right)^2}$$

$$\min\frac{t+3}{t\left(1-t\right)}=\frac{t+3}{t\left(1-t\right)}\Big|_{-3+2\sqrt{3}}=\frac{\sqrt{3}\left(2\sqrt{3}+3\right)\left(2+\sqrt{3}\right)}{3}$$

thus

$$\mathbf{E}e^{s(X-\mathbf{E}X)}\leq e^{\frac{\sqrt{3}\left(2\sqrt{3}+3\right)\left(2+\sqrt{3}\right)}{6}\frac{s^2}{c}}\leq e^{\frac{7s^2}{c}}$$

this is a subgaussian tail bound. □